Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 392741, 24 pages
doi:10.1155/2011/392741

Research Article

Strong Convergence of a New Iterative Method for Infinite Family of Generalized Equilibrium and Fixed-Point Problems of Nonexpansive Mappings in Hilbert Spaces

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Received 15 October 2010; Accepted 18 November 2010

Academic Editor: Qamrul Hasan Ansari

We introduce an iterative algorithm for finding a common element of the set of solutions of an infinite family of equilibrium problems and the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space. We prove some strong convergence theorems for the proposed iterative scheme to a fixed point of the family of nonexpansive mappings, which is the unique solution of a variational inequality. As an application, we use the result of this paper to solve a multiobjective optimization problem. Our result extends and improves the ones of Colao et al. (2008) and some others.

1. Introduction

Let H be a real Hilbert space and T be a mapping of H into itself. T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{1.1}$$

If there exists a point $u \in H$ such that $Tu = u$, then the point u is called a fixed point of T. The set of fixed points of T is denoted by $F(T)$. It is well known that $F(T)$ is closed convex and also nonempty if T has a bounded trajectory (see [1]).

Let $f : H \to H$ be a mapping. If there exists a constant $0 \le \kappa < 1$ such that

$$\|fx - fy\| \le \kappa \|x - y\|, \quad \forall x, y \in H, \tag{1.2}$$

then f is called a contraction with the constant κ. Recall that an operator $A : H \to H$ is called to be strongly positive with coefficient $\overline{\gamma} > 0$ if

$$\langle Ax, x\rangle \ge \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.3}$$

Let $u \in H$ be a fixed point, A be a strongly positive linear bounded operator on H and $\{T\}_{n=1}^{N}$ be a finite family of nonexpansive mappings of H into itself such that $F = \bigcap_{n=1}^{N} F(T_n) \neq \emptyset$.

In 2003, Xu [2] introduced the following iterative scheme:

$$x_{n+1} = (I - \epsilon_{n+1}A)T_{n+1}x_n + \epsilon_{n+1}u, \quad \forall n \ge 1, \tag{1.4}$$

where I is the identical mapping on H and $T_n = T_{n \bmod N}$, and proved some strong convergence theorems for the iterative scheme to the solution of the quadratic minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle \tag{1.5}$$

under suitable hypotheses on ϵ_n and the additional hypothesis:

$$F = F(T_1T_2\cdots T_N) = F(T_NT_1\cdots T_{N-1}) = \cdots = F(T_2T_3\cdots T_NT_1). \tag{1.6}$$

Recently, Marino and Xu [3] introduced a new iterative scheme from an arbitrary point $x_0 \in H$ by the viscosity approximation method as follows:

$$x_{n+1} = \epsilon_n\gamma f(x_n) + (I - \epsilon_n A)Tx_n, \quad \forall n \ge 1, \tag{1.7}$$

and prove that the scheme strongly converges to the unique solution x^* of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^*\rangle \ge 0, \quad \forall x \in F(T), \tag{1.8}$$

which is the optimality condition for the minimization problem:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.9}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$).

Let $\{T_n\}_{n=1}^{N}$ be a finite family of nonexpansive mappings of H into itself. In 2007, Yao [4] defined the mappings

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n \equiv U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})I \end{aligned} \tag{1.10}$$

and, by extending (1.10), proposed the iterative scheme:

$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1-\beta)I - \epsilon_n A\big)W_n x_n, \quad \forall n \geq 1. \tag{1.11}$$

Then he proved that the iterative scheme (1.10) strongly converges to the unique solution x^* of the variational inequality:

$$\big\langle (A-\gamma f)x^*, x - x^* \big\rangle \geq 0, \quad \forall x \in F, \tag{1.12}$$

where $F = \bigcap_{n=1}^{N} F(T_n)$, which is the optimality condition for the minimization problem:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.13}$$

where h is a potential function for γf (However, Colao et al. pointed out in [5] that there is a gap in Yao's proof).

Let C be a nonempty closed convex subset of H and $G : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for the function G is to determine the equilibrium points, that is, the set

$$EP(G) = \big\{x \in C : G(x,y) \geq 0,\ \forall y \in C\big\}. \tag{1.14}$$

Let $A : C \to H$ be a nonlinear mapping. Let $EP(G,A)$ denote the set of all solutions to the following equilibrium problem:

$$EP(\mathrm{c}G, A) = \big\{x \in C : G(x,y) + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\big\}. \tag{1.15}$$

In the case of $A \equiv 0$, $EP(G,A)$ is deduced to EP. In the case of $G \equiv 0$, $EP(G,A)$ is also denoted by $\mathrm{VI}(C,A)$.

In 2007, S. Takahashi and W. Takahashi [6] introduced a viscosity approximation method for finding a common element of $EP(G)$ and $F(T)$ from an arbitrary initial element $x_1 \in H$

$$\begin{aligned} &G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \epsilon_n f(x_n) + (1-\epsilon_n)Tu_n, \quad \forall n \geq 1, \end{aligned} \tag{1.16}$$

and proved that, under certain appropriate conditions over ϵ_n and r_n, the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to $z = P_{F(T)\cap EP(G)} f(z)$.

By combing the schemes (1.7) and (1.16), Plubtieg and Punpaeng [7] proposed the following algorithm:

$$\begin{aligned} &G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \epsilon_n \gamma f(x_n) + (I-\epsilon_n A)Tu_n, \quad \forall n \geq 1, \end{aligned} \tag{1.17}$$

and proved that the iterative schemes $\{x_n\}$ and $\{u_n\}$ converge strongly to the unique solution z of the variational inequality:

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(T) \cap EP(G), \tag{1.18}$$

which is the optimality condition for the minimization problem:

$$\min_{x\in F(T)\cap EP(G)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.19}$$

where h is a potential function for γf.

Very recently, for finding a common element of the set of a finite family of nonexpansive mappings and the set of solutions of an equilibrium problem, by combining the schemes (1.11) and (1.17), Colao et al. [5] proposed the following explicit scheme:

$$\begin{aligned} &G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)W_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{1.20}$$

and proved under some certain hypotheses that both sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $x^* \in F$ which is an equilibrium point for G and is the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in F \cap EP(G), \tag{1.21}$$

where $F = \bigcap_{n=1}^{N} F(T_n)$.

The equilibrium problems have been considered by many authors; see, for example, [6, 8–19] and the reference therein. But, in these references, the authors only considered at most finite family of equilibrium problems and few of authors investigate the infinite family of equilibrium problems in a Hilbert space or Banach space. In this paper, we consider a new iterative scheme for obtaining a common element in the solution set of an infinite family of generalized equilibrium problems and in the common fixed-point set of a finite family of nonexpansive mappings in a Hilbert space. Let $\{T_n\}_{n=1}^{N}$ $(N \geq 1)$ be a finite family of nonexpansive mappings of H into itself, be $\{G_n\} : C \times C \to \mathbb{R}$ be an infinite family of bifunctions, and be $\{A_n\} : C \to H$ be an infinite family of k_n-inverse-strongly monotone mappings. Let $\{r_n\}$ be a sequence such that $r_n \subset (r, 2k_n)$ with $r > 0$ for each $n \geq 1$. Define the mapping $T_{r_i} : H \to C$ by

$$T_{r_i}(x) = \left\{ z \in C : G_i(z, y) + \frac{1}{r_i}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}, \quad x \in H,\ i \geq 1. \tag{1.22}$$

Assume that $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap_{i=1}^{\infty} EP(G_i, A_i) \neq \emptyset$. For an arbitrary initial point $x_1 \in H$, we define the iterative scheme $\{x_n\}$ by

$$\begin{gathered} z_n = \alpha_n x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_{r_i}(I - r_i A_i) x_n, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \delta_n B x_n + (I - \delta_n B - \epsilon_n A) W_n z_n, \quad \forall n \geq 1, \end{gathered} \tag{1.23}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$, $\{\epsilon_n\}$ and $\{\delta_n\}$ are three sequences in (0,1), A and B are both strongly positive linear bounded operators on H, W_n is defined by (1.10), and prove that, under some certain appropriate hypotheses on the control sequences, the sequence $\{x_n\}$ strongly converges to a point $x^* \in \Omega$, which is the unique solution of the variational inequality:

$$\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{1.24}$$

If $A_i \equiv A_0$, $G_i \equiv G$ and $r_i \equiv r$, then (1.23) is reduced to the iterative scheme:

$$\begin{gathered} z_n = \alpha_n x_n + (1 - \alpha_n) T_r(I - r A_0) x_n, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \delta_n B x_n + (I - \delta_n B - \epsilon_n A) W_n z_n, \quad \forall n \geq 1. \end{gathered} \tag{1.25}$$

The proof method of the main result of this paper is different with ones of others in the literatures and our result extends and improves the ones of Colao et al. [5] and some others.

2. Preliminaries

Let C be a closed convex subset of a Hilbert space H. For any point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

Then P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies the following:

$$\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Let A be a mapping from C into H, then A is called monotone if

$$\langle x - y, Ax - Ay \rangle \ge 0 \tag{2.3}$$

for all $x, y \in C$. However, A is called an α-inverse-strongly monotone mapping if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \ge \alpha \|Ax - Ay\|^2 \tag{2.4}$$

for all $x, y \in C$. Let I denote the identity mapping of H, then for all $x, y \in C$ and $\lambda > 0$, one has [20]

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \tag{2.5}$$

Hence, if $\lambda \in (0, 2\alpha]$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

If there exists $u \in C$ such that

$$\langle v - u, Au \rangle \ge 0 \tag{2.6}$$

for all $v \in C$, then u is called the solution of this variational inequality. The set of all solutions of the variational inequality is denoted by $\mathrm{VI}(C, A)$.

In this paper, we need the following lemmas.

Lemma 2.1 (see [21]). *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C. \tag{2.7}$$

Lemma 2.2 (see [22]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \eta_n) s_n + \eta_n \tau_n + \xi_n, \quad \forall n \geq 0, \tag{2.8}$$

where $\{\eta_n\}$, $\{\tau_n\}$, and $\{\xi_n\}$ satisfy the conditions:

(1) $\{\eta_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \eta_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - \eta_n) = 0$;

(2) $\limsup_{n\to\infty} \tau_n \leq 0$;

(3) $\xi_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \xi_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Let H be a Hilbert space. For all $x, y \in H$, the following equality holds:

$$\|x + y\|^2 = \|x\|^2 + 2\langle y, x + y \rangle - \|y\|^2. \tag{2.9}$$

Therefore, the following lemma naturally holds.

Lemma 2.3. *Let H be a real Hilbert space. The following identity holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \tag{2.10}$$

Lemma 2.4 (see [3]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.5 (see [2]). *Assume that $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad \forall n \geq 0, \tag{2.11}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} (\delta_n / \gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (see [23]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies the following:*

(A1) $G(x,x) = 0$ *for all* $x \in C$;

(A2) G *is monotone, that is,* $G(x,y) + G(y,x) \leq 0$ *for all* $x, y \in C$;

(A3) *For each* $x, y, z \in C$,

$$\lim_{t \downarrow 0} G\big(tz + (1-t)x, y\big) \leq G\big(x,y\big); \tag{2.12}$$

(A4) *For each* $x \in C$, $y \mapsto G(x,y)$ *is convex and lower semicontinuous.*

For $x \in H$ *and* $r > 0$, *define a mapping* $T_r : H \to C$ *by*

$$T_r(x) = \left\{ z \in C : G\big(z,y\big) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}. \tag{2.13}$$

Then T_r *is well defined and the following hold:*

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\left\|T_r x - T_r y\right\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.14}$$

(3) $F(T_r) = \mathrm{EP}(G)$;

(4) $\mathrm{EP}(G)$ *is closed and convex.*

It is easy to see that if there exists some point $v \in C$ such that $v = T_r(I - rA)v$, where $A : C \to H$ is an α-inverse strongly monotone mapping, then $v \in EP(G, A)$. In fact, since $v = T_r(I - rA)v$, one has

$$G\big(v,y\big) + \frac{1}{r}\langle y - v, v - (I - rA)v\rangle \geq 0, \quad \forall y \in C, \tag{2.15}$$

that is,

$$G\big(v,y\big) + \langle y - v, Av\rangle \geq 0, \quad \forall y \in C. \tag{2.16}$$

Hence, $v \in EP(G, A)$.

Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$

for each $i = 1, 2, \ldots, N$. Define a mapping W of C into itself as follows:

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) I, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) I, \\ W = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) I. \end{aligned} \tag{2.17}$$

Such a mapping W is called the *W-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ (see [5, 24, 25]).

Lemma 2.7 (see [26]). *Let C be a nonempty closed convex subset of a Banach space. Let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for each $i = 1, 2, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let W be the W-mapping of C generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then $F(W) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.8 (see [5]). *Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\{\lambda_{n,i}\}_{i=1}^N$ be sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$ for each $i = 1, 2, \ldots, N$. Moreover, for each $n \in \mathbb{N}$, let W and W_n be the W-mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for all $x \in C$, it follows that*

$$\lim_{n \to \infty} \|W_n x - W x\| = 0. \tag{2.18}$$

3. Main Results

Now, we give our main results in this paper.

Theorem 3.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $f : H \to H$ be a contraction with coefficient $0 < \kappa < 1$, $A, B : H \to H$ be strongly positive linear bounded self-adjoint operators with coefficients $\overline{\gamma} > 0$ and $\overline{\beta} > 2\|B\| + \|B\|^2$, respectively, $\{T_n\}_{n=1}^N : H \to H$ $(N \geq 1)$ be a finite family of nonexpansive mappings, $\{G_n\} : C \times C \to \mathbb{R}$ be an infinite family of bifunctions satisfying $(A1)$–$(A4)$, and $\{A_n\} : C \to H$ be an infinite family of inverse-strongly monotone mappings with constants $\{k_n\}$ such that $\Omega = (\bigcap_{i=1}^N F(T_i)) \cap (\bigcap_{i=1}^\infty EP(G_i, A_i)) \neq \emptyset$. Let $\{\epsilon_n\}$ and $\{\delta_n\}$ be two sequences in $(0,1)$, $\{\lambda_{n,i}\}_{i=1}^N$ be asequence in $[a,b]$ with $0 < a \leq b < 1$, $\{r_n\}$ be a sequence in $(r, 2k_n)$ with $r > 0$ and $\{\alpha_n\}$ be a strictly decreasing sequence $[0,1)$. Set $\alpha_0 = 1$. Take a fixed number $\gamma > 0$ with $0 < \overline{\gamma} - \gamma\kappa < 1$. Assume that*

- (E1) $\lim_{n \to \infty} \epsilon_n = 0$ and $\sum_{n=1}^\infty \epsilon_n = \infty$;
- (E2) $\lim_{n \to \infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ for each $i = 1, 2, \ldots, N$;
- (E3) $0 \leq \delta_n + \epsilon_n \leq 1$ for all $n \geq 1$;
- (E4) $\{\delta_n\} \subset [0, \min\{c, 1/\beta,\ (2\|B\| + \|B\|^2 - \overline{\beta} + \sqrt{(\overline{\beta} - \|B\|^2 - 2\|B\|)^2 + 8\overline{\beta}\|B\|})/4\overline{\beta}\|B\|\})$ *with* $c < 1$;
- (E5) $\sum_{n=1}^\infty |\epsilon_{n+1} - \epsilon_n| < \infty$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n| < \infty$.

Then the sequence $\{x_n\}$ *defined by* (1.23) *converges strongly to* $x^* \in \Omega$, *which is the unique solution of the variational inequality:* (1.24), *that is,*

$$x^* = P_{\Omega}(I - (A - \gamma f))x^*. \tag{3.1}$$

Proof. Since $\epsilon_n \to 0$ as $n \to \infty$ by the condition (E1), we may assume, without loss of generality, that $\epsilon_n < (1 - \delta_n\|B\|)\|A\|^{-1}$ for all $n \geq 1$. Noting that A and B are both the linear bounded self-adjoint operators, one has

$$\begin{aligned} \|A\| &= \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}, \\ \|B\| &= \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}. \end{aligned} \tag{3.2}$$

Observing that

$$\begin{aligned} \langle (I - \delta_n B - \epsilon_n A)x, x\rangle &= 1 - \delta_n\langle Bx, x\rangle - \epsilon_n\langle Ax, x\rangle \\ &\geq 1 - \delta_n\|B\| - \epsilon_n\|A\| \\ &\geq 0, \end{aligned} \tag{3.3}$$

we obtain that $I - \delta_n B - \epsilon_n A$ is positive for all $n \geq 1$. It follows that

$$\begin{aligned} \|I - \delta_n B - \epsilon_n A\| &= \sup\{\langle (I - \delta_n B - \epsilon_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \langle (\delta_n B + \epsilon_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \delta_n\overline{\beta} - \epsilon_n\overline{\gamma}. \end{aligned} \tag{3.4}$$

For each $n \geq 1$, define a quadratic function $f(\delta_n)$ in δ_n as follows:

$$f(\delta_n) = 2\overline{\beta}\|B\|\delta_n^2 + \left(\overline{\beta} - \|B\|^2 - 2\|B\|\right)\delta_n. \tag{3.5}$$

Note that

$$f(0) = f\left(\frac{2\|B\| - \overline{\beta} + \|B\|^2}{2\overline{\beta}\|B\|}\right) = 0, \tag{3.6}$$

$$f\left(\frac{2\|B\| + \|B\|^2 - \overline{\beta} + \sqrt{\left(\overline{\beta} - \|B\|^2 - 2\|B\|\right)^2 + 8\overline{\beta}\|B\|}}{4\overline{\beta}\|B\|}\right) = 1. \tag{3.7}$$

Hence, for each δ_n satisfying the condition (E4), one has

$$0 < 2\overline{\beta}\|B\|\delta_n^2 + \left(\overline{\beta} - \|B\|^2 - 2\|B\|\right)\delta_n < 1. \tag{3.8}$$

Moreover, it follows from (3.7), $f(1/\|B\|) > 1$ and (E4) that

$$\delta_n < \frac{1}{\|B\|}, \quad \forall n \geq 1. \tag{3.9}$$

Next, we proceed the proof with following steps.

Step 1. $\{x_n\}$ is bounded.

Let $p \in \Omega$. Lemma 2.6 shows that every T_{r_i} is firmly nonexpansive and hence nonexpansive. Since $r < r_i < 2k_i$, $I - r_i A_i$ is nonexpansive for each $i \geq 1$. Therefore, $T_{r_i}(I - r_i A_i)$ is nonexpansive for each $i \geq 1$. Noting that $\{\alpha_n\}$ is strictly decreasing, $\alpha_0 = 1$, we have

$$\begin{aligned}\|z_n - p\| &= \left\|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\big(T_{r_i}(I - r_i A_i)x_n - T_{r_i}(I - r_i A_i)p\big)\right\| \\ &\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|T_{r_i}(I - r_i A_i)x_n - T_{r_i}(I - r_i A_i)p\| \\ &\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - p\| \\ &= \|x_n - p\|\end{aligned} \tag{3.10}$$

and hence

$$\|W_n z_n - p\| = \|W_n z_n - W_n p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.11}$$

Then, from (3.4) and (3.11), it follows that (noting that B is linear and $\overline{\beta} > 2\|B\| + \|B\|^2 \Rightarrow \overline{\beta} > \|B\|$)

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad= \|\epsilon_n(\gamma f(x_n) - Ap) + \delta_n(Bx_n - Bp) + (I - \delta_n B - \epsilon_n A)(W_n z_n - p)\| \\ &\quad\leq \epsilon_n\|\gamma f(x_n) - Ap\| + \delta_n\|B(x_n - p)\| + \|I - \delta_n B - \epsilon_n A\|\|W_n z_n - p\| \\ &\quad\leq \epsilon_n\gamma\kappa\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| + \delta_n\|B\|\|x_n - p\| + \left(1 - \delta_n\overline{\beta} - \epsilon_n\overline{\gamma}\right)\|W_n z_n - p\| \\ &\quad\leq \epsilon_n\gamma\kappa\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\| + \delta_n\|B\|\|x_n - p\| + \left(1 - \delta_n\overline{\beta} - \epsilon_n\overline{\gamma}\right)\|x_n - p\| \\ &\quad\leq (1 - \epsilon_n(\overline{\gamma} - \gamma\kappa))\|x_n - p\| + \epsilon_n\|\gamma f(p) - Ap\|.\end{aligned} \tag{3.12}$$

It follows from $\epsilon_n \in (0,1)$ and $0 < \overline{\gamma} - \gamma\kappa < 1$ that $0 < \epsilon_n(\overline{\gamma} - \gamma\kappa) < 1$. Therefore, by the simple induction, we have

$$\|x_n - p\| \leq \max\left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma\kappa} \right\}, \quad \forall n \geq 1, \tag{3.13}$$

which shows that $\{x_n\}$ is bounded, so is $\{z_n\}$.

Step 2. $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

First, we prove

$$\lim_{n\to\infty} \|W_{n+1}z_n - W_n z_n\| = 0. \tag{3.14}$$

Let $i \in \{0, 1, \ldots, N-2\}$ and set

$$M_1 = \sup_n \left\{ \|z_n\| + \|T_1 z_n\| + \sum_{i=2}^{N} \|T_i U_{n,i-1} z_n\| \right\} < \infty. \tag{3.15}$$

It follows from the definition of W_n that

$$\begin{aligned}
&\|U_{n+1,N-i}z_n - U_{n,N-i}z_n\| \\
&\quad = \|\lambda_{n+1,N-i}T_{N-i}U_{n+1,N-i-1}z_n + (1-\lambda_{n+1,N-i})z_n - \lambda_{n,N-i}T_{N-i}U_{n,N-i-1}z_n - (1-\lambda_{n,N-i})z_n\| \\
&\quad \leq \lambda_{n+1,N-i}\|T_{N-i}U_{n+1,N-i-1}z_n - T_{N-i}U_{n,N-i-1}z_n\| \\
&\qquad + |\lambda_{n+1,N-i} - \lambda_{n,N-i}|\|T_{N-i}U_{n,N-i-1}z_n\| + |\lambda_{n+1,N-i} - \lambda_{n,N-i}|\|z_n\| \\
&\quad \leq \|U_{n+1,N-i-1}z_n - U_{n,N-i-1}z_n\| + (\|z_n\| + \|T_{N-i}U_{n,N-i-1}z_n\|)|\lambda_{n+1,N-i} - \lambda_{n,N-i}| \\
&\quad \leq \|U_{n+1,N-i-1}z_n - U_{n,N-i-1}z_n\| + M_1|\lambda_{n+1,N-i} - \lambda_{n,N-i}|
\end{aligned} \tag{3.16}$$

for each $i \in \{0, 1, \ldots, N-2\}$. Thus, using the above recursive inequalities repeatedly, we have

$$\begin{aligned}
\|W_{n+1}z_n - W_n z_n\| &= \|U_{n+1,N}z_n - U_{n,N}z_n\| \\
&\leq M_1 \sum_{i=2}^{N} |\lambda_{n+1,i} - \lambda_{n,i}| + |\lambda_{n+1,1} - \lambda_{n,1}|(\|z_n\| + \|T_1 z_n\|) \\
&\leq M_1 \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned} \tag{3.17}$$

Also, we have

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \Bigg\| \alpha_n x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_{r_i}(I - r_i A_i) x_n - \alpha_{n-1} x_{n-1} \\
&\qquad - \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_{r_i}(I - r_i A_i) x_{n-1} + (\alpha_{n-1} - \alpha_n) T_{r_n}(I - r_n A_n) x_{n-1} \Bigg\| \\
&\le \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|x_n - x_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|T_{r_n}(I - r_n A_n) x_{n-1}\| \\
&= \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|T_{r_n}(I - r_n A_n) x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| L,
\end{aligned} \tag{3.18}$$

where $L = \sup\{\|x_{n-1}\| + \|T_{r_n}(I - r_n A_n) x_{n-1}\|\}$.

Next, we prove $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Observe (noting that B is linear) that

$$\begin{aligned}
x_{n+1} - x_n &= \epsilon_n \gamma \big(f(x_n) - f(x_{n-1}) \big) + \epsilon_n \gamma f(x_{n-1}) + \delta_n B(x_n - x_{n-1}) + \delta_n B x_{n-1} \\
&\quad + (I - \delta_n B - \epsilon_n A)(W_n z_n - W_n z_{n-1}) + (I - \delta_n B - \epsilon_n A) W_n z_{n-1} \\
&\quad - \epsilon_{n-1} \gamma f(x_{n-1}) - \delta_{n-1} B x_{n-1} - (I - \delta_{n-1} B - \epsilon_{n-1} A) W_{n-1} z_{n-1} \\
&= \epsilon_n \gamma \big(f(x_n) - f(x_{n-1}) \big) + \delta_n B(x_n - x_{n-1}) + (I - \delta_n B - \epsilon_n A)(W_n z_n - W_n z_{n-1}) \\
&\quad + (\epsilon_n - \epsilon_{n-1}) \gamma f(x_{n-1}) + (\delta_n - \delta_{n-1}) B x_{n-1} + (W_n z_{n-1} - W_{n-1} z_{n-1}) \\
&\quad + (\delta_{n-1} B W_{n-1} z_{n-1} - \delta_n B W_n z_{n-1}) + (\epsilon_{n-1} A W_{n-1} z_{n-1} - \epsilon_n A W_n z_{n-1}) \\
&= \epsilon_n \gamma \big(f(x_n) - f(x_{n-1}) \big) + \delta_n B(x_n - x_{n-1}) + (I - \delta_n B - \epsilon_n A)(W_n z_n - W_n z_{n-1}) \\
&\quad + (\epsilon_n - \epsilon_{n-1}) \gamma f(x_{n-1}) + (\delta_n - \delta_{n-1}) B x_{n-1} + (W_n z_{n-1} - W_{n-1} z_{n-1}) \\
&\quad + (\delta_{n-1} - \delta_n) B W_{n-1} z_{n-1} + \delta_n B(W_{n-1} z_{n-1} - W_n z_{n-1}) \\
&\quad + (\epsilon_{n-1} - \epsilon_n) A W_{n-1} z_{n-1} + \epsilon_n A(W_{n-1} z_{n-1} - W_n z_{n-1}).
\end{aligned} \tag{3.19}$$

Hence, by (3.4) and (3.18), we get

$$
\begin{aligned}
\|x_{n+1} - x_n\| \le{}& \epsilon_n\gamma\kappa\|x_n - x_{n-1}\| + \delta_n\|B\|\|x_n - x_{n-1}\| + \left(1 - \delta_n\overline{\beta} - \epsilon_n\overline{\gamma}\right)\|z_n - z_{n-1}\| \\
&+ |\epsilon_n - \epsilon_{n-1}|\gamma\left\|f(x_{n-1})\right\| + |\delta_n - \delta_{n-1}|\|Bx_{n-1}\| + \|W_n z_{n-1} - W_{n-1}z_{n-1}\| \\
&+ |\delta_{n-1} - \delta_n|\|BW_{n-1}z_{n-1}\| + \delta_n\|B\|\|W_{n-1}z_{n-1} - W_n z_{n-1}\| \\
&+ |\epsilon_{n-1} - \epsilon_n|\|AW_{n-1}z_{n-1}\| + \epsilon_n\|A\|\|W_{n-1}z_{n-1} - W_n z_{n-1}\| \\
\le{}& \epsilon_n\gamma\kappa\|x_n - x_{n-1}\| + \delta_n\|B\|\|x_n - x_{n-1}\| \\
&+ \left(1 - \delta_n\overline{\beta} - \epsilon_n\overline{\gamma}\right)(\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|L) \\
&+ |\epsilon_n - \epsilon_{n-1}|\gamma\left\|f(x_{n-1})\right\| + |\delta_n - \delta_{n-1}|\|Bx_{n-1}\| + \|W_n z_{n-1} - W_{n-1}z_{n-1}\| \\
&+ |\delta_{n-1} - \delta_n|\|BW_{n-1}z_{n-1}\| + \delta_n\|B\|\|W_{n-1}z_{n-1} - W_n z_{n-1}\| \\
&+ |\epsilon_{n-1} - \epsilon_n|\|AW_{n-1}z_{n-1}\| + \epsilon_n\|A\|\|W_{n-1}z_{n-1} - W_n z_{n-1}\| \\
\le{}& \left[1 - \left(\delta_n\left(\overline{\beta} - \|B\|\right) + \epsilon_n(\overline{\gamma} - \gamma\kappa)\right)\right]\|x_n - x_{n-1}\| + L|\alpha_n - \alpha_{n-1}| + 2|\epsilon_{n-1} - \epsilon_n|M_2 \\
&+ 2|\delta_{n-1} - \delta_n|M_2 + (1 + \delta_n\|B\| + \epsilon_n\|A\|)\|W_{n-1}z_{n-1} - W_n z_{n-1}\|,
\end{aligned}
\tag{3.20}
$$

where $M_2 = \sup_n\{\gamma\|f(x_{n-1})\| + \|Bx_{n-1}\| + \|BW_{n-1}z_{n-1}\| + \|AW_{n-1}z_{n-1}\|\}$.

Set $M_3 = \min\{\overline{\beta} - \|B\|, \overline{\gamma} - \gamma\kappa\}$. It follows from $0 \le \overline{\gamma} - \gamma\kappa < 1$ and $\overline{\beta} > \|B\|$ (due to $\overline{\beta} > 2\|B\| + \|B\|^2$) that $0 \le M_2 < 1$. Thus we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \le{}& [1 - (\delta_n + \epsilon_n)M_3]\|x_n - x_{n-1}\| + (\delta_n + \epsilon_n)M_3 \\
&\times \left(\frac{1}{(\delta_n + \epsilon_n)M_3} + \frac{\delta_n\|B\|}{(\delta_n + \epsilon_n)M_3} + \frac{\epsilon_n\|A\|}{(\delta_n + \epsilon_n)M_3}\right) \\
&\times \|W_{n-1}z_{n-1} - W_n z_{n-1}\| + L|\alpha_n - \alpha_{n-1}| + 2|\epsilon_{n-1} - \epsilon_n|M_2 + 2|\delta_{n-1} - \delta_n|M_2.
\end{aligned}
\tag{3.21}
$$

Set

$$
\begin{aligned}
\eta_n &= (\delta_n + \epsilon_n)M_3, \\
\tau_n &= \left(\frac{1}{(\delta_n + \epsilon_n)M_3} + \frac{\delta_n\|B\|}{(\delta_n + \epsilon_n)M_3} + \frac{\epsilon_n\|A\|}{(\delta_n + \epsilon_n)M_3}\right)\|W_{n-1}z_{n-1} - W_n z_{n-1}\|, \\
\xi_n &= L|\alpha_n - \alpha_{n-1}| + 2|\epsilon_{n-1} - \epsilon_n|M_2 + 2|\delta_{n-1} - \delta_n|M_2.
\end{aligned}
\tag{3.22}
$$

Then it follows from (3.21) that

$$\|x_{n+1} - x_n\| \leq (1 - \eta_n)\|x_n - x_{n-1}\| + \eta_n \tau_n + \xi_n. \tag{3.23}$$

It follows from the assumption condition (E1), (E3), (E5), and (3.14) that

$$\eta_n \subset [0,1], \qquad \sum_{n=1}^{\infty} \eta_n = \infty, \qquad \lim_{n \to \infty} \tau_n = 0, \qquad \sum_{n=1}^{\infty} \xi_n < \infty. \tag{3.24}$$

By applying Lemma 2.2 to (3.23), we obtain $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3. $x_n - W_n z_n \to 0$ as $n \to \infty$.

For all $n \geq 1$, we have

$$\begin{aligned}
\|x_n - W_n z_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n z_n\| \\
&= \|x_n - x_{n+1}\| + \left\|\epsilon_n \gamma f(x_n) + \delta_n B x_n + [I - \delta_n B - \epsilon_n A] W_n z_n - W_n z_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \epsilon_n \left\|\gamma f(x_n) - A W_n z_n\right\| + \delta_n \|B x_n - B W_n z_n\| \\
&\leq \|x_n - x_{n+1}\| + \epsilon_n \left\|\gamma f(x_n) - A W_n z_n\right\| + \delta_n \|B\| \|x_n - W_n z_n\|
\end{aligned} \tag{3.25}$$

and hence (noting (3.9))

$$\|x_n - W_n z_n\| \leq \frac{1}{1 - \delta_n \|B\|} \|x_n - x_{n+1}\| + \frac{\epsilon_n}{1 - \delta_n} \left[\gamma \left\|f(x_n)\right\| + \|A W_n z_n\|\right]. \tag{3.26}$$

It follows from the assumption conditions (E1), (E2), and Step 2 that

$$x_n - W_n z_n \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.27}$$

Step 4. $x_n - z_n \to 0$ as $n \to \infty$.

Notice that, for any $x \in \Omega$,

$$\begin{aligned}
\|z_n - x\|^2 &= \left\|\alpha_n(x_n - x) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(T_{r_i}(I - r_i A_i)x_n - T_{r_i}(I - r_i A_i)x)\right\|^2 \\
&\le \alpha_n\|x_n - x\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|(I - r_i A_i)x_n - (I - r_i A_i)x\|^2 \\
&\le \alpha_n\|x_n - x\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\left[\|x_n - x\|^2 + r_i(r_i - 2k_i)\|A_i x_n - A_i x\|^2\right] \\
&= \|x_n - x\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) r_i(r_i - 2k_i)\|A_i x_n - A_i x\|^2.
\end{aligned} \tag{3.28}$$

Let $y_n = \gamma f(x_n) - AW_n z_n$ and $\lambda = \sup\{\|\gamma f(x_n) - AW_n z_n\| : n \ge 1\}$. By using (3.8), (3.9), (3.28), Lemmas 2.3, and 2.4, we have (noting that $\delta_n < 1/\overline{\beta}$)

$$\begin{aligned}
\|x_{n+1} - x\|^2 &= \left\|(I - \delta_n B)(W_n z_n - x) + \delta_n(Bx_n - Bx) + \epsilon_n(\gamma f(x_n) - AW_n z_n)\right\|^2 \\
&\le \|(I - \delta_n B)(W_n z_n - x) + \delta_n(Bx_n - Bx)\|^2 + 2\epsilon_n\langle y_n, x_{n+1} - x\rangle \\
&= \|(I - \delta_n B)(W_n z_n - W_n x) + \delta_n B(x_n - x)\|^2 + 2\epsilon_n\langle y_n, x_{n+1} - x\rangle \\
&\le \left(1 - \delta_n\overline{\beta}\right)\|z_n - x\|^2 + \delta_n\|B\|^2\|x_n - x\|^2 + 2\delta_n\left(1 - \delta_n\overline{\beta}\right)\|B\|\|x_n - x\|^2 + 2\lambda^2\epsilon_n \\
&\le \left(1 - \delta_n\overline{\beta}\right)\left[\|x_n - x\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) r_i(r_i - 2k_i)\|A_i x_n - A_i x\|^2\right] \\
&\quad + \delta_n\|B\|^2\|x_n - x\|^2 + 2\delta_n\left(1 - \delta_n\overline{\beta}\right)\|B\|\|x_n - x\|^2 + 2\lambda^2\epsilon_n \\
&= \left[1 - \left(\delta_n\overline{\beta} - \delta_n\|B\|^2 - 2\delta_n\left(1 - \delta_n\overline{\beta}\right)\|B\|\right)\right]\|x_n - x\|^2 \\
&\quad + \left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) r_i(r_i - 2k_i)\|A_i x_n - A_i x\|^2 + 2\lambda^2\epsilon_n \\
&\le \|x_n - x\|^2 + \left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) r_i(r_i - 2k_i)\|A_i x_n - A_i x\|^2 + 2\lambda^2\epsilon_n.
\end{aligned} \tag{3.29}$$

This shows that

$$\begin{aligned}
&\left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) r_i(2k_i - r_i)\|A_i x_n - A_i x\|^2 \\
&\quad \le \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + 2\lambda^2\epsilon_n \\
&\quad \le (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| + 2\lambda^2\epsilon_n
\end{aligned} \tag{3.30}$$

and hence, for each $i \geq 1$,

$$\begin{aligned}
&\left(1 - \delta_n \overline{\beta}\right)(\alpha_{i-1} - \alpha_i) r_i (2k_i - r_i) \|A_i x_n - A_i x\|^2 \\
&\quad \leq \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + 2\lambda^2 \epsilon_n \\
&\quad \leq (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| + 2\lambda^2 \epsilon_n.
\end{aligned} \tag{3.31}$$

Since $\delta_n \rightarrow 0$, $\|x_n - x_{n+1}\| \rightarrow 0$ and $\alpha_{i-1} - \alpha_i > 0$, we have

$$\lim_{n \to \infty} \|A_i x_n - A_i x\| = 0, \quad i \geq 1. \tag{3.32}$$

Now, for $x \in \Omega$, we have, from Lemma 2.2,

$$\begin{aligned}
&\|T_{r_i}(I - r_i A_i)x_n - x\|^2 \\
&\quad = \|T_{r_i}(I - r_i A_i)x_n - T_{r_i}(I - r_i A_i)x\|^2 \\
&\quad \leq \langle T_{r_i}(I - r_i A_i)x_n - T_{r_i}(I - r_i A_i)x, (I - r_i A_i)x_n - (I - r_i A_i)x \rangle \\
&\quad = \langle T_{r_i}(I - r_i A_i)x_n - x, x_n - x \rangle + r_i \langle T_{r_i}(I - r_i A_i)x_n - x, A_i x - A_i x_n \rangle \\
&\quad = \frac{1}{2}\left\{ \|T_{r_i}(I - r_i A_i)x_n - x\|^2 + \|x_n - x\|^2 - \|x_n - T_{r_i}(I - r_i A_i)x_n\|^2 \right\} \\
&\qquad + r_i \langle T_{r_i}(I - r_i A_i)x_n - x, A_i x - A_i x_n \rangle
\end{aligned} \tag{3.33}$$

and hence

$$\begin{aligned}
\|T_{r_i}(I - r_i A_i)x_n - x\|^2 \leq{}& \|x_n - x\|^2 - \|x_n - T_{r_i}(I - r_i A_i)x_n\|^2 \\
&+ 2r_i \langle T_{r_i}(I - r_i A_i)x_n - x, A_i x - A_i x_n \rangle.
\end{aligned} \tag{3.34}$$

Therefore,

$$\begin{aligned}
\|z_n - x\|^2 &\leq \alpha_n \|x_n - x\|^2 + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|T_{r_i}(I - r_i A_i)x_n - x\|^2 \\
&\leq \alpha_n \|x_n - x\|^2 + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \Big[\|x_n - x\|^2 - \|x_n - T_{r_i}(I - r_i A_i)x_n\|^2 \\
&\qquad + 2r_i \langle T_{r_i}(I - r_i A_i)x_n - x, A_i x - A_i x_n \rangle \Big] \\
&= \|x_n - x\|^2 - \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|x_n - T_{r_i}(I - r_i A_i)x_n\|^2 \\
&\quad + 2\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) r_i \langle T_{r_i}(I - r_i A_i)x_n - x, A_i x - A_i x_n \rangle.
\end{aligned} \tag{3.35}$$

By using (3.8), (3.9), (3.35), Lemmas 2.3 and 2.4, we have (noting that $\delta_n < 1/\overline{\beta}$)

$$\begin{aligned}
&\|x_{n+1} - x\|^2 \\
&\quad = \left\|(I - \delta_n B)(W_n z_n - x) + \delta_n(Bx_n - Bx) + \epsilon_n(\gamma f(x_n) - AW_n z_n)\right\|^2 \\
&\quad \le \|(I - \delta_n B)(W_n z_n - x) + \delta_n(Bx_n - Bx)\|^2 + 2\epsilon_n\langle y_n, x_{n+1} - x\rangle \\
&\quad = \|(I - \delta_n B)(W_n z_n - W_n x) + \delta_n B(x_n - x)\|^2 + 2\epsilon_n\langle y_n, x_{n+1} - x\rangle \\
&\quad \le \left(1 - \delta_n\overline{\beta}\right)\|z_n - x\|^2 + \delta_n\|B\|^2\|x_n - x\|^2 + 2\delta_n\left(1 - \delta_n\overline{\beta}\right)\|B\|\|x_n - x\|^2 + 2\lambda^2\epsilon_n \\
&\quad \le \left(1 - \delta_n\overline{\beta}\right)\Bigg[\|x_n - x\|^2 - \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - T_{r_i}(I - r_iA_i)x_n\|^2 \\
&\qquad\qquad + 2\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)r_i\langle T_{r_i}(I - r_iA_i)x_n - x, A_ix - A_ix_n\rangle\Bigg] \\
&\qquad + \delta_n\|B\|^2\|x_n - x\|^2 + 2\delta_n\left(1 - \delta_n\overline{\beta}\right)\|B\|\|x_n - x\|^2 + 2\lambda^2\epsilon_n \\
&\quad = \left[1 - \left(\delta_n\overline{\beta} - \delta_n\|B\|^2 - 2\delta_n\left(1 - \delta_n\overline{\beta}\right)\|B\|\right)\right]\|x_n - x\|^2 \\
&\qquad - \left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - T_{r_i}(I - r_iA_i)x_n\|^2 \\
&\qquad + 2\left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)r_i\langle T_{r_i}(I - r_iA_i)x_n - x, A_ix - A_ix_n\rangle + 2\lambda^2\epsilon_n \\
&\quad \le \|x_n - x\|^2 - \left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - T_{r_i}(I - r_iA_i)x_n\|^2 \\
&\qquad + 2\left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)r_i\langle T_{r_i}(I - r_iA_i)x_n - x, A_ix - A_ix_n\rangle + 2\lambda^2\epsilon_n
\end{aligned} \tag{3.36}$$

and hence

$$\begin{aligned}
&\left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - T_{r_i}(I - r_iA_i)x_n\|^2 \\
&\quad \le \|x_n - x\|^2 - \left\|x_{n+1} - p\right\|^2 + 2\left(1 - \delta_n\overline{\beta}\right) \\
&\qquad \times \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)r_i\langle T_{r_i}(I - r_iA_i)x_n - x, A_ix - A_ix_n\rangle + 2\lambda^2\epsilon_n \\
&\quad \le (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| \\
&\qquad + 2\left(1 - \delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)r_i\langle T_{r_i}(I - r_iA_i)x_n - x, A_ix - A_ix_n\rangle + 2\lambda^2\epsilon_n.
\end{aligned} \tag{3.37}$$

This shows that for, each $i \geq 1$,

$$\begin{aligned}
&\left(1-\delta_n\overline{\beta}\right)(\alpha_{i-1}-\alpha_i)\|x_n-T_{r_i}(I-r_iA_i)x_n\|^2 \\
&\quad \leq (\|x_n-x\|+\|x_{n+1}-x\|)\|x_n-x_{n+1}\| \\
&\qquad + 2\left(1-\delta_n\overline{\beta}\right)\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)r_i\langle T_{r_i}(I-r_iA_i)x_n-x, A_ix-A_ix_n\rangle + 2\lambda^2\epsilon_n.
\end{aligned} \tag{3.38}$$

Since $\{\alpha_n\}$ is strictly decreasing, $\delta_n \to 0$, $\epsilon_n \to 0$, $A_ix_n - A_ix \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we have, for each $i \geq 1$,

$$\|x_n - T_{r_i}(I-r_iA_i)x_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.39}$$

Now, from $z_n - x_n = \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_{r_i}x_n - x_n)$ we get

$$\|z_n - x_n\| \leq \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_{r_i}x_n - x_n\|. \tag{3.40}$$

Since $\|x_n - T_{r_i}x_n\| \to 0$ and $0 < \alpha_{i-1}-\alpha_i$ for each $i \geq 1$, one has

$$\|z_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.41}$$

Step 5. $\limsup_{n\to\infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle \leq 0$.

To prove this, we pick a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle = \lim_{j\to\infty}\left\langle(\gamma f - A)x^*, x_{n_j} - x^*\right\rangle. \tag{3.42}$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \widehat{x}$. Obviously, to prove Step 5, we only need to prove that $\widehat{x} \in \Omega$.

Indeed, for each $i \geq 1$, since $x_n - T_{r_i}(I-r_iA_i)x_n \to 0$, $x_{n_j} \to \widehat{x}$ and $T_{r_i}(I-r_iA_i)$ is nonexpansive, by demiclosed principle of nonexpansive mapping we have

$$\widehat{x} \in F(T_{r_i}(I-r_iA_i)) = EP(G_i, A_i), \quad i \geq 1. \tag{3.43}$$

Assume that $\lambda_{n_m,k} \to \lambda_k \in (0,1)$ for each $k = 1,2,\ldots,N$. Let W be the W-mapping generated by $T_1,\ldots,T_N$ and $\lambda_1,\ldots,\lambda_N$. Then, by Lemma 2.8, we have

$$W_{n_m}x \longrightarrow Wx, \quad \forall x \in H. \tag{3.44}$$

Moreover, it follows from Lemma 2.7 that $\bigcap_{n=1}^{N} F(T_i) = F(W)$. Assume that $\widehat{x} \notin F(W)$. Then $\widehat{x} \neq W\widehat{x}$. Since $\widehat{x} \in F(T_{r_i}(I - r_i A_i))$ for each $i \geq 1$, by Step 3, (3.44) and Opial's property of the Hilbert space H, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|x_{n_m} - \widehat{x}\| \\
&\quad < \liminf_{n\to\infty} \|x_{n_m} - W\widehat{x}\| \\
&\quad \leq \liminf_{n\to\infty} (\|x_{n_m} - W_{n_m} z_{n_m}\| + \|W_{n_m} z_{n_m} - W_{n_m}\widehat{x}\| + \|W_{n_m}\widehat{x} - W\widehat{x}\|) \\
&\quad \leq \liminf_{n\to\infty} (\|x_{n_m} - W_{n_m} z_{n_m}\| + \|z_{n_m} - \widehat{x}\| + \|W_{n_m}\widehat{x} - W\widehat{x}\|) \\
&\quad \leq \liminf_{n\to\infty} (\|x_{n_m} - W_{n_m} z_{n_m}\| + \|z_{n_m} - x_{n_m}\| + \|x_{n_m} - T_{r_i}(I - r_i A_i) x_{n_m}\| \\
&\qquad + \|\widehat{x} - T_{r_i}(I - r_i A_i) x_{n_m}\| + \|W_{n_m}\widehat{x} - W\widehat{x}\|) \\
&\quad \leq \liminf_{n\to\infty} (\|x_{n_m} - W_{n_m} z_{n_m}\| + \|z_{n_m} - x_{n_m}\| + \|x_{n_m} - T_{r_i}(I - r_i A_i) x_{n_m}\| \\
&\qquad + \|T_{r_i}(I - r_i A_i)\widehat{x} - T_{r_i}(I - r_i A_i) x_{n_m}\| + \|W_{n_m}\widehat{x} - W\widehat{x}\|) \\
&\quad \leq \liminf_{n\to\infty} \|x_{n_m} - \widehat{x}\|,
\end{aligned} \tag{3.45}$$

which is a contradiction. Therefore, $\widehat{x} \in F(W)$. Hence, $\widehat{x} \in \Omega = (\bigcap_{i=1}^{N} F(T_i)) \cap (\bigcap_{i=1}^{\infty} EP(G_i, A_i))$.

Step 6. The sequence $\{x_n\}$ strongly converges to some point $x^* \in H$.

By using Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|(I - \delta_n B - \epsilon_n A)(W_n z_n - x^*) + \delta_n (B x_n - B x^*) + \epsilon_n \left(\gamma f(x_n) - A x^*\right)\right\|^2 \\
&\leq \|(I - \delta_n B - \epsilon_n A)(W_n z_n - x^*) + \delta_n (B x_n - B x^*)\|^2 \\
&\quad + 2\epsilon_n \langle \gamma f(x_n) - A x^*, x_{n+1} - x^* \rangle \\
&\leq \left[(1 - \delta_n \|B\| - \epsilon_n \overline{\gamma})\|z_n - x^*\| + \delta_n \|B\| \|x_n - x^*\|\right]^2 \\
&\quad + 2\epsilon_n \langle \gamma f(x_n) - A x^*, x_{n+1} - x^* \rangle \\
&\leq \left[(1 - \delta_n \|B\| - \epsilon_n \overline{\gamma})\|x_n - x^*\| + \delta_n \|B\| \|x_n - x^*\|\right]^2 \\
&\quad + 2\epsilon_n \langle \gamma f(x_n) - A x^*, x_{n+1} - x^* \rangle \\
&= \left[(1 - \epsilon_n \overline{\gamma})\|x_n - x^*\|\right]^2 + 2\epsilon_n \langle \gamma f(x_n) - A x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \epsilon_n \overline{\gamma})^2 \|x_n - x^*\|^2 + 2\epsilon_n \gamma \kappa \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + 2\epsilon_n \langle \gamma f(x^*) - A x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \epsilon_n \overline{\gamma})^2 \|x_n - x^*\|^2 + \epsilon_n \gamma \kappa \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + 2\epsilon_n \langle \gamma f(x^*) - A x^*, x_{n+1} - x^* \rangle,
\end{aligned} \tag{3.46}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \frac{(1-\epsilon_n\overline{\gamma})^2 + \epsilon_n\gamma\kappa}{1-\epsilon_n\gamma\kappa}\|x_n - x^*\|^2 + \frac{2\epsilon_n}{1-\epsilon_n\gamma\kappa}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\quad = \frac{(1-2\epsilon_n\overline{\gamma} + \epsilon_n\gamma\kappa)}{1-\epsilon_n\gamma\kappa}\|x_n - x^*\|^2 + \frac{\epsilon_n^2\overline{\gamma}^2}{1-\epsilon_n\gamma\kappa}\|x_n - x^*\|^2 + \frac{2\epsilon_n}{1-\epsilon_n\gamma\kappa}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\quad \le \left[1 - \frac{2\epsilon_n(\overline{\gamma} - \kappa\gamma)}{1-\epsilon_n\gamma\kappa}\right]\|x_n - x^*\|^2 \\
&\qquad + \frac{2\epsilon_n(\overline{\gamma} - \kappa\gamma)}{1-\epsilon_n\gamma\kappa}\left[\frac{1}{\overline{\gamma} - \kappa\gamma}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle + \frac{\epsilon_n\overline{\gamma}^2}{2(\overline{\gamma} - \kappa\gamma)}M'\right],
\end{aligned} \tag{3.47}$$

where M' is an appropriate constant such that $M' = \sup_{n\ge 1}\{\|x_n - x^*\|\}$. Put

$$\begin{aligned}
s_n &= \frac{2\epsilon_n(\overline{\gamma} - \kappa\gamma)}{1-\epsilon_n\kappa\gamma}, \\
t_n &= \frac{1}{\overline{\gamma} - \kappa\gamma}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle + \frac{\epsilon_n\overline{\gamma}^2}{2(\overline{\gamma} - \kappa\gamma)}M'.
\end{aligned} \tag{3.48}$$

Then we have

$$\|x_{n+1} - x^*\|^2 \le (1 - s_n)\|x_n - x^*\| + s_n t_n. \tag{3.49}$$

It follows from the assumption condition (E1) and (3.42) that

$$\lim_{n\to\infty} s_n = 0, \qquad \sum_{n=1}^{\infty} s_n = \infty, \qquad \limsup_{n\to\infty} t_n \le 0. \tag{3.50}$$

Thus, applying Lemma 2.5 to (3.49), it follows that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

By Theorem 3.1, we have the following direct corollaries.

Corollary 3.2. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $f : H \to H$ be a contraction with coefficient $0 < \kappa < 1$, $A : H \to H$ be strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$, $\{T_n\}_{n=1}^N : H \to H$ $(N \ge 1)$ be a finite family of nonexpansive mappings, $G : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4), and $A_0 : C \to H$ be an α-inverse strongly monotone mapping such that $\Omega = \bigcap_{i=1}^N F(T_i) \cap EP(G, A) \ne \emptyset$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be two sequences in $(0, 1)$, $\{\lambda_{n,i}\}_{i=1}^N$ be a sequence in $[a, b]$ with $0 < a \le b < 1$, r be a number in $(0, 2\alpha)$, and $\{\alpha_n\}$ be a sequence $[0, 1)$. Take a fixed number $\gamma > 0$ with $0 < \overline{\gamma} - \gamma\kappa < 1$. Assume that*

(E1) $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\varepsilon_n = \infty$;

(E2) $\lim_{n\to\infty}|\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for each* $i = 1, 2, \ldots, N$;

(E3) $0 \le \delta_n + \epsilon_n \le 1$ *for all* $n \ge 1$;

(E4) $\{\delta_n\} \subset [0, \min\{c, 1/\overline{\beta},\ (2\|B\| + \|B\|^2 - \overline{\beta} + \sqrt{(\overline{\beta} - \|B\|^2 - 2\|B\|)^2 + 8\overline{\beta}\|B\|})/4\overline{\beta}\|B\|\})$ *with* $c < 1$;

(E5) $\sum_{n=1}^{\infty}|\epsilon_{n+1} - \epsilon_n| < \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$.

Then the sequence $\{x_n\}$ *defined by* (1.25) *converges strongly to* $x^* \in \Omega$, *which is the unique solution of the variational inequality:*

$$x^* = P_\Omega(I - (A - \gamma f))x^*. \tag{3.51}$$

Remark 3.3. In the proof process of Theorem 3.1, we do not use Suzuki's Lemma (see [27]), which was used by many others for obtaining $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ (see [4, 5, 28]). The proof method of $\widehat{x} \in \bigcap_{i=1}^{\infty} EP(G_i, A_i)$ is simple and different with ones of others.

4. Applications for Multiobjective Optimization Problem

In this section, we study a kind of multiobjective optimization problem by using the result of this paper. That is, we will give an iterative algorithm of solution for the following multiobjective optimization problem with the nonempty set of solutions:

$$\begin{matrix} \min\ h_1(x), \\ \min\ h_2(x), \end{matrix} \quad x \in C, \tag{4.1}$$

where $h_1(x)$ and $h_2(x)$ are both the convex and lower semicontinuous functions defined on a closed convex subset of C of a Hilbert space H.

We denote by A the set of solutions of the problem (4.1) and assume that $A \ne \emptyset$. Also, we denote the sets of solutions of the following two optimization problems by A_1 and A_2, respectively,

$$\min\ h_1(x), \quad x \in C, \tag{4.2}$$

and

$$\min\ h_2(x), \quad x \in C. \tag{4.3}$$

Note that, if we find a solution $x \in A_1 \cap A_2$, then one must have $x \in A$ obviously.

Now, let G_1 and G_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ defined by

$$G_1(x, y) = h_1(y) - h_1(x), \quad G_2(x, y) = h_2(y) - h_2(x), \quad \forall (x, y) \in C \times C, \tag{4.4}$$

respectively. It is easy to see that $EP(G_1) = A_1$ and $EP(G_2) = A_2$, where $EP(G_i)$ denotes the set of solutions of the equilibrium problem:

$$G_i(x, y) \geq 0, \quad \forall y \in C, \ i = 1, 2, \tag{4.5}$$

respectively. In addition, it is easy to see that G_1 and G_2 satisfy the conditions (A1)–(A4). Let $\{\alpha_n\}$ be a sequence in (0,1) and $r_1, r_2 \in (0, 1)$. Define a sequence $\{x_n\}$ by

$$\begin{aligned} x_1 &\in H, \\ z_n &= \alpha_n T_{r_1} x_n + (1 - \alpha_n) T_{r_2} x_n, \\ x_{n+1} &= \epsilon_n \gamma f(x_n) + (1 - \epsilon_n) z_n, \quad \forall n \geq 1. \end{aligned} \tag{4.6}$$

By Theorem 3.1 with $A = I$, $N = 1$, $T_1 = I$ and $\delta_n = 0$ for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a solution $x^* = P_{A_1 \cap A_2} \gamma f(x^*)$, which is a solution of the multiobjective optimization problem (4.1).

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